INTERNATIONAL BACCALAUREATE
BACCALAURÉAT INTERNATIONAL
BACHILLERATO INTERNACIONAL

MATHEMATICS
HIGHER LEVEL
PAPER 2

Thursday 4 May 2000 (morning)

3 hours

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all five questions from Section A and one question from Section B.
- Unless otherwise stated in the question, all numerical answers must be given exactly or to three significant figures, as appropriate.

- Write the make and model of your calculator on the front cover of your answer booklets *e.g.* Casio *fx-7400G*, Sharp EL-9400, Texas Instruments TI-80.

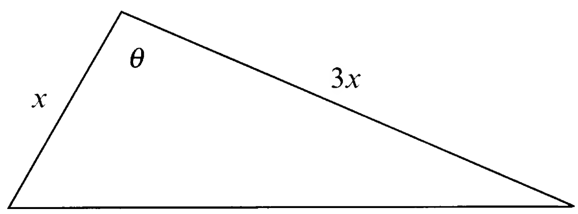
You are advised to start each new question on a new page. A correct answer with **no** *indication of the method used will usually receive* **no** *marks. You are therefore advised to show your working. In particular, where graphs from a graphic display calculator are being used to find solutions, you should sketch these graphs as part of your answer.*

SECTION A

Answer all ***five*** *questions from this section.*

1. *[Maximum mark: 10]*

The area of the triangle shown below is 2.21 cm^2. The length of the shortest side is x cm and the other two sides are $3x$ cm and $(x + 3)$ cm .



(a) Using the formula for the area of the triangle, write down an expression for $\sin \theta$ in terms of x . *[2 marks]*

(b) Using the cosine rule, write down and simplify an expression for $\cos \theta$ in terms of x . *[2 marks]*

(c) (i) Using your answers to parts (a) and (b), show that,

$$\left(\frac{3x^2 - 2x - 3}{2x^2}\right)^2 = 1 - \left(\frac{4.42}{3x^2}\right)^2$$ *[1 mark]*

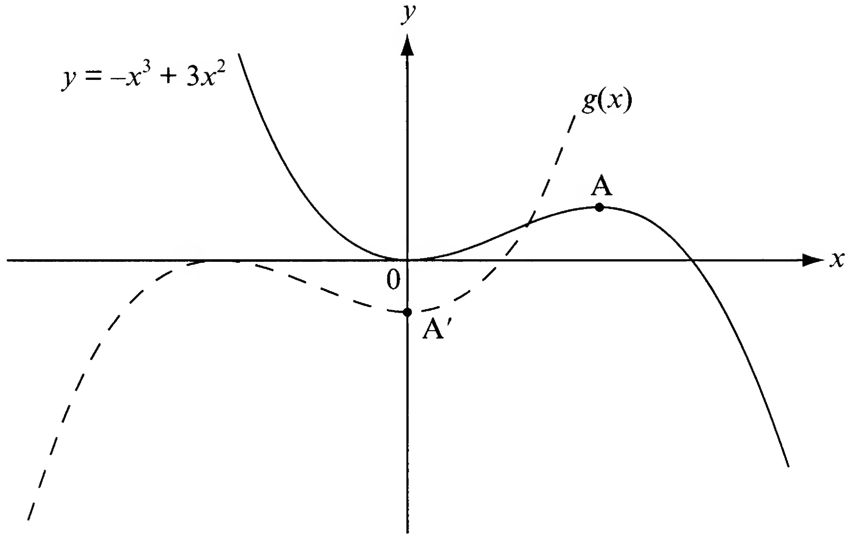
(ii) Hence find

(a) the possible values of x; *[2 marks]*

(b) the corresponding values of θ , **in radians**, using your answer to part (b) above. *[3 marks]*

2. *[Maximum mark: 14]*

The diagram below shows the graphs of $y = -x^3 + 3x^2$ and $y = g(x)$, where $g(x)$ is a polynomial of degree 3 .



(a) If $g(-2) = 0$, $g(0) = -4$, $g'(-2) = 0$, and $g'(0) = (0)$ show that $g(x) = x^3 + 3x^2 - 4$. *[6 marks]*

The graph of $y = -x^3 + 3x^2$ is reflected in the y-axis, then translated using the vector $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ to give the graph of $y = h(x)$.

(b) Write $h(x)$ in the form $h(x) = ax^3 + bx^2 + cx + d$. *[5 marks]*

The graph of $y = x^3 + 3x^2 - 4$ is obtained by applying a composition of two transformations to the graph of $y = -x^3 + 3x^2$.

(c) State the two transformations whose composition maps the graph of $y = -x^3 + 3x^2$ onto the graph of $y = x^3 + 3x^2 - 4$ and also maps point A onto point A′ . *[3 marks]*

3. *[Maximum mark: 16]*

Let $f(x) = \ln | x^5 - 3x^2 |$, $-0{\cdot}5 < x < 2$, $x \neq a$, $x \neq b$; (a , b are values of x for which $f(x)$ is not defined).

(a) (i) Sketch the graph of $f(x)$, indicating on your sketch the number of zeros of $f(x)$. Show also the position of any asymptotes . *[2 marks]*

(ii) Find all the zeros of $f(x)$, (that is, solve $f(x) = 0$) . *[3 marks]*

(b) Find the **exact** values of a and b . *[3 marks]*

(c) Find $f'(x)$, and indicate clearly where $f'(x)$ is not defined. *[3 marks]*

(d) Find the **exact** value of the x-coordinate of the local maximum of $f(x)$, for $0 < x < 1.5$. (You may assume that there is no point of inflexion.) *[3 marks]*

(e) **Write down** the definite integral that represents the area of the region **enclosed** by $f(x)$ and the x-axis. (Do **not** evaluate the integral.) *[2 marks]*

4. *[Maximum mark: 12]*

A machine is set to produce bags of salt, whose weights are distributed normally, with a mean of 110 g and standard deviation of 1.142 g . If the weight of a bag of salt is less than 108 g , the bag is rejected. With these settings, 4% of the bags are rejected.

The settings of the machine are altered and it is found that 7% of the bags are rejected.

(a) (i) If the mean has not changed, find the new standard deviation, **correct to three decimal places**. *[4 marks]*

The machine is adjusted to operate with this new value of the standard deviation.

(ii) Find the value, **correct to two decimal places**, at which the mean should be set so that only 4% of the bags are rejected. *[4 marks]*

(b) With the new settings from part (a), it is found that 80% of the bags of salt have a weight which lies between A g and B g, where A and B are symmetric about the mean. Find the values of A and B, giving your answers **correct to two decimal places**. *[4 marks]*

5. *[Maximum mark: 18]*

(i) Differentiate from first principles $f(x) = \cos x$. *[8 marks]*

(ii) Prove by mathematical induction that $\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}$, for all positive integer values of n. *[10 marks]*

SECTION B

*Answer **one** question from this section.*

Statistics

6. *[Maximum mark: 30]*

A computer manufacturing company buys large quantities of hard discs from several suppliers. Hard disc quality is checked by a process called RTT which gives results on a continuous scale from 0 to 100 . Based on previous experience the company assumes that the results are normally distributed with a mean of 68 and standard deviation of 3 .

Shipments arrive from suppliers on a daily basis. A sample of 10 hard discs is taken from each shipment at random and tested. If the mean of the sample is more than 67 , the shipment is accepted, otherwise it is rejected.

(a) What is the probability that a hard disc selected at random has a result less than 67 ? *[2 marks]*

(b) Find the probability that a shipment is rejected. *[4 marks]*

(c) There is a \$1000 penalty each day that a shipment is rejected. A particular supplier's hard discs have a mean of 67.5 and a standard deviation of 2.8 .

(i) What is the probability that this supplier's shipment is accepted? *[3 marks]*

(ii) What is the expected penalty per 6-day week for this supplier? *[6 marks]*

(d) The company's own production of hard discs has a mean of 68 and a standard deviation of 3 . However, to keep the production within the acceptable limits, the company samples 10 hard discs every hour and examines whether the sample is accepted or rejected. During a particular hour, the following results were recorded for a sample that was randomly chosen for testing:

68.1747, 68.0473, 66.3189, 66.2735, 66.957, 66.9738,
66.1438, 67.0744, 66.1875, 67.8804

At the 5% level of significance determine whether the sample meets the company's standard for acceptance. *[7 marks]*

(This question continues on the following page.)

(Question 6 continued)

(e) Every week, the company randomly selects the test results of 1000 hard discs and checks if these results come from a normal distribution (with a mean of 68 and standard deviation of 3) or not. The following table gives the results, R, for one such test.

Results	Frequency
$56 \le R < 59$	5
$59 \le R < 62$	17
$62 \le R < 65$	146
$65 \le R < 68$	333
$68 \le R < 71$	360
$71 \le R < 74$	113
$R \ge 74$	26

Check, at the 5% level of significance, whether the above data comes from a normal population with a mean of 68 and standard deviation of 3. *[8 marks]*

Sets, Relations, and Groups

7. *[Maximum mark: 30]*

(i) Let X and Y be two non-empty sets.

(a) Define the operation $X \bullet Y$ by $X \bullet Y = (X \cap Y) \cup (X' \cap Y')$. Prove that $(X \bullet Y)' = (X \cup Y) \cap (X' \cup Y')$. *[3 marks]*

(b) Let $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = n + 1$, for all $n \in \mathbb{N}$. Determine if f is an injection, a surjection, or a bijection. Give reasons for your answer. *[3 marks]*

(c) Let $h : X \to Y$, and let R be an equivalence relation on Y. $y_1 \, R \, y_2$ denotes that two elements y_1 and y_2 of Y are related.

Define a relation S on X by the following:

For all a, $b \in X$, $a \, S \, b$ if and only if $h(a) \, R \, h(b)$.

Determine if S is an equivalence relation on X. *[4 marks]*

(ii) (a) Let f_1, f_2, f_3, f_4 be functions defined on $\mathbb{Q} - \{0\}$, the set of rational numbers excluding zero, such that $f_1(z) = z$, $f_2(z) = -z$, $f_3(z) = \frac{1}{z}$, and $f_4(z) = -\frac{1}{z}$, where $z \in \mathbb{Q} - \{0\}$.

Let $T = \{f_1, f_2, f_3, f_4\}$. Define $\circ$ as the composition of functions *i.e.* $(f_1 \circ f_2)(z) = f_1(f_2(z))$. Prove that $(T, \circ)$ is an Abelian group. *[6 marks]*

(b) Let $G = \{1, 3, 5, 7\}$ and $(G, \diamond)$ be the multiplicative group under the binary operation $\diamond$, multiplication modulo 8. Prove that the two groups $(T, \circ)$ and $(G, \diamond)$ are isomorphic. *[5 marks]*

(iii) Let a, b and p be elements of a group $(H, *)$ with an identity element e.

(a) If element a has order n and element a^{-1} has order m, then prove that $m = n$. *[5 marks]*

(b) If $b = p^{-1} * a * p$, prove, by mathematical induction, that $b^m = p^{-1} * a^m * p$, where $m = 1, 2, \ldots$. *[4 marks]*

Discrete Mathematics

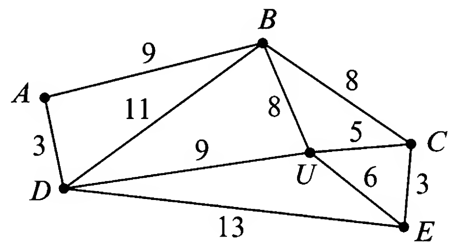
8. *[Maximum marks: 30]*

(i) A chemical manufacturer has to transport six chemical products from a factory to a processing plant by rail. Some of the products cannot be taken in the same railroad car because of the possibility of their mixing together and creating a violent reaction if an accident happens. The six products are labelled as A, B, C, D, E, and F. It is known that A cannot be kept in the same railroad car as B, C, or D; B cannot be kept in the same railroad car as C or E; C cannot be kept with D; and E cannot be kept with F.

(a) Draw a graph where each vertex represents a product and the edges join pairs of products which cannot be in the same railroad car. *[2 marks]*

(b) Find the chromatic number of this graph and show a possible arrangement for transporting the six products. *[4 marks]*

(ii) Let G be the graph given below:



(a) Has G got an Eulerian circuit? Give a reason for your answer. *[2 marks]*

(b) What is the adjacency matrix of the graph G? Determine how many walks of length 2 are there from vertex A to vertex C. *[4 marks]*

(c) Use Kruskal's algorithm to find the minimum spanning tree for graph G. *[5 marks]*

(iii) Find the solution of the difference equation

$$y_{n+2} = y_{n+1} + y_n \, , \; y_0 = 4 \, , \text{ and } y_1 = 3 \, .$$ *[6 marks]*

(iv) (a) State the well-ordering principle for the set of positive integers. *[2 marks]*

(b) Extend this notion to any set where elements can be ordered. *[2 marks]*

(c) By giving reasons, carefully, establish whether $\mathbb{Z}$, the set of integers, is well-ordered. *[3 marks]*

Analysis and Approximations

9. *[Maximum mark: 30]*

(i) Let $f(x) = x^7 + 5x + 1$, $-2 \le x \le 2$.

(a) Use the Newton-Raphson method with an initial value $x_0 = -0.5$ to obtain a real root (zero) of f correct to 8 decimal places. Explain how your result for the third iteration has been obtained. *[4 marks]*

(b) Apply fixed point iteration with $x_0 = -0.5$ to calculate three iterates to find a solution of $f(x) = 0$. From your calculation, determine if the fixed point iteration will give a real root (zero) of f . Give a reason for your answer. *[4 marks]*

(c) Using Rolle's theorem prove that the equation $f(x) = 0$ has exactly one real root. *[4 marks]*

(ii) (a) Find the Maclaurin series of the function $g(x) = \sin x^2$ using the series expansion of $\sin x$, *i.e.* $\sin x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}$. *[1 mark]*

(b) Using the Maclaurin series of $g(x) = \sin x^2$ evaluate the definite integral

$$\int_0^1 \sin x^2 \, dx$$

correct to four decimal places. *[5 marks]*

(iii) (a) Use the ratio test to calculate the radius of convergence of the power series $\sum_{n=1}^{\infty} \dfrac{(x-5)^n}{n^{\frac{3}{2}}}$. *[3 marks]*

(b) Using your result from part (a), determine **all points** x where the power series given in (a) converges. *[5 marks]*

(iv) Using the mean value theorem prove that

$$|\sin x \cos x - \sin y \cos y| \le |x - y| .$$ *[4 marks]*

Euclidean Geometry and Conic Sections

10. *[Maximum mark: 30]*

(i) Let e_1 and e_2 be the eccentricities of the hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$, respectively. Prove that

$$\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1$$ *[3 marks]*

(ii) Let F_1 and F_2 be the foci of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and P be any point on the hyperbola with coordinates (x_0, y_0).

(a) Let (PM) be the tangent to the hyperbola at P, with M being on the x-axis between F_1 and F_2. Find the coordinates of the point M. *[4 marks]*

(b) Find the lengths PF_1, PF_2, MF_1, MF_2 in terms of a, x_0 and e. *[5 marks]*

(c) Using an appropriate theorem from euclidean geometry, prove that [PM] is the angle bisector of $F_1\widehat{P}F_2$, stating which theorem you are using. *[4 marks]*

(iii) If ABCD is a cyclic quadrilateral, then prove Ptolemy's theorem:

$$AB \times CD + BC \times DA = AC \times BD$$ *[5 marks]*

(iv) In a triangle ABC, $A\widehat{C}B > A\widehat{B}C$. Let M and N be points on [AC] and [AB], respectively, such that [CN] and [BM] are the angle bisectors of $A\widehat{C}B$ and $A\widehat{B}C$ respectively. Let M′ be the point on [BM] which is nearer to the point M and is such that $M'\widehat{C}N = \frac{1}{2}(A\widehat{B}C)$.

(a) Prove that BNM′C is a cyclic quadrilateral. *[4 marks]*

(b) Prove that BM′ > CN. *[5 marks]*